距離空間におけるCOHOMOLOGY次元について(一般・幾何学的位相と関連する諸問題)

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距離空間における COHOMOLOGY 次元について

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1. INTRODUCTION AND PRELIMINARY

In the last ten years, cohomological dimension theory has striking development. A motivation of the development is surely the Edwards-Walsh theorem, [24], as follows:

1.1. Theorem. *Every compact metric space X of cohomological dimension $c\text{-}\dim_{\mathbf{Z}} X \leq n$ (integer coefficient) is the image of a cell-like map $f\colon Z \to X$ from a compact metric space Z of $\dim Z \leq n$.*

Not only the result but also techniques of the proof gave an important influence to the development. After them, L. R. Rubin and P. J. Schapiro [22] showed the noncompact version of the Edwards-Walsh theorem and S. Mardešić and L. R. Rubin [17] gave the nonmetrizable version. On the other hand, A. N. Dranishnikov, [5] and [6], characterized cohomological dimension with respect to $\mathbf{Z}_p$ by the Edwards-Walsh's way and showed the Edwards-Walsh-like theorem:

1.2. Theorem. *Every compact metric space X of cohomological dimension with respect to $\mathbf{Z}_p$, $c\text{-}\dim_{\mathbf{Z}_p} X \leq n$, is the image of a map $f\colon Z \to X$ from a compact metric space Z of $\dim Z \leq n$ whose fibers are acyclic modulo p.*

Motivated above results and Mardešić's characterization of $c\text{-}\dim_{\mathbf{Z}} X \leq n$, we will show a characterization of $c\text{-}\dim_{\mathbf{Z}_p} X \leq n$ for noncompact case. Using the characterization, we will give the existence of an acyclic resolution modulo p. In fact, our characterization suggests a dimension-like function, called approximable dimension, and can obtain the following more general results.

1.3. Theorem. *Let X be a metrizable space having approximable dimension with respect to an arbitrary coefficients $G \leq n$. Then there exists a map $f\colon Z \to X$ from a metrizable space Z of $\dim Z \leq n$ and $w(Z) \leq w(X)$ onto X such that $H^*(f^{-1}(x); G) = 0$ for all $x \in X$.*

As its consequence, we have noncompact versions of Theorems 1.1 and 1.2. We may call such a mapping f an acyclic resolution of X (with respect to G), specially, in the case of $G = \mathbf{Z}_p$, an acyclic resolution of X modulo p. Finally we will note that there exists a compact metric space X of $c\text{-}\dim_{\mathbf{Q}} X = 1$ which does not admit an acyclic resolution with respect to $\mathbf{Q}$ [11,12]. Thereby we can see that approximable dimension is different from cohomological dimension and Theorem 1.3 is a good property obtained from approximable dimension.

In this paper, we mean the definition of cohomological dimension as follows: the *cohomological dimension of a space X with respect to a coefficient group G is less than and equal to n*, denoted by $c\text{-}\dim_G X \leq n$, provided that every map $f\colon A \to K(G,n)$ of a closed subset A of X into an Eilenberg-MacLane space $K(G,n)$ of type (G,n) admits a continuous extension over X (c.f. [10]). The dimension of a space X means the *covering dimension* of X and denotes by $\dim X$. $\mathbf{Z}$ is the additive group of all integers and for each prime number p, $\mathbf{Z}_p$ is the cyclic group of order p.

By a polyhedron we mean the space $|K|$ of a simplicial complex K with the *Whitehead topology*. In section 5, the topology of $|K|$ may be generated by a uniformity [Appendix, 22].

If v is a vertex of a simplicial complex K, let $\mathrm{st}(v,K)$ be the open star of v in $|K|$ and $\bar{\mathrm{st}}(v,K)$ be the closed star of v in $|K|$. If $A \subseteq |K|$, then we define $\mathrm{st}(A,K) = \bigcup\{\mathrm{Int}\,\sigma : \sigma \in K, \sigma \cap A \neq \emptyset\}$ and $\bar{\mathrm{st}}(A,K) = \bigcup\{\sigma : \sigma \in K, \sigma \cap A \neq \emptyset\}$. The symbol $\mathrm{Sd}_j\, K$ means the j-th barycentric subdivision of K. We define the symbols $\mathcal{S}_i$ and $\bar{\mathcal{S}}_i$ for a simplicial complex K_i with an index to be the cover $\{\mathrm{st}(v,K_i) : v \in K_i^{(0)}\}$ and the cover $\{\bar{\mathrm{st}}(v,K_i) : v \in K_i^{(0)}\}$, respectively.

We use the symbol $\prec$ both to mean 'refine' for covers and 'subdivides' for subdivisions of a complex. The symbol $\prec^*$ is used for star refines.

Let $\mathcal{U}$ be an open cover of a space X. Then for $U \in \mathcal{U}$,

$$\mathrm{st}(U,\mathcal{U}) = \mathrm{st}^1(U,\mathcal{U}) = \bigcup\{U' : U' \in \mathcal{U}, U' \cap U \neq \emptyset\},$$
$$\mathrm{st}^{j+1}(U,\mathcal{U}) = \bigcup\{U' : U' \in \mathcal{U}, U' \cap \mathrm{st}^j(U,\mathcal{U}) \neq \emptyset\}.$$

By $\mathrm{st}^j(\mathcal{U})$ we mean the cover $\{\mathrm{st}^j(U,\mathcal{U}) : U \in \mathcal{U}\}$. If f and g are maps from a space Z to a space X, $(f,g) \leq \mathcal{U}$ means that for each $z \in Z$, there exists $U \in \mathcal{U}$ with $f(z), g(z) \in U$. If X is a metric space with a metric d, we write $(f,g) \leq \varepsilon$ instead of $(f,g) \leq \mathcal{U}_\varepsilon$, where $\mathcal{U}_\varepsilon$ is the cover whose consists of all $\varepsilon/2$-neighborhoods in X. By the symbol $\mathcal{N}(\mathcal{U})$ we mean the nerve of the cover $\mathcal{U}$. For covers $\mathcal{U}$, $\mathcal{V}$, the symbol $\mathcal{U} \wedge \mathcal{V}$ is used for the following cover $\{U \cap V, U, V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

2. Edwards-Walsh complexes

In the latter section, we need Edwards-Walsh complexes for arbitrary simplicial complexes.

2.1. Lemma. *Let $|L|$ be a simplicial complex with the Whitehead topology, p be a prime number and n be a natural number. Then there exists a combinatorial map (i.e.*

$\pi_L^{-1}(L')$ *is a subcomplex of* $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$ *if* L' *is a subcomplex of* L*)* $\psi_L\colon \mathrm{EW}_{\mathbf{Z}_p}(L,n) \to |L|$ *such that*

(i) *for* $\sigma \in L$ *with* $\dim\sigma \geq n+1$, $\psi_L^{-1}(\sigma) \in K(\oplus_1^{r_\sigma}\mathbf{Z}_p, n)$, *where* $r_\sigma = \operatorname{rank}\pi_n(\sigma^{(n)})$,
(ii) *for* $\sigma \in L$ *with* $\dim\sigma \leq n$, $\psi_L^{-1}(\sigma) = \sigma$,
(iii) $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$ *is a CW-complex,*
(iv) $\psi_L^{-1}(\sigma)$ *is a subcomplex of* $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$ *with respect to the triangulation in (3),*
(v) $\psi_L^{-1}(\sigma)^{(k)}$ *is a finite CW-complex for* $k \geq n$,
(vi) *for any subcomplex* L' *of* L *and map* $f\colon |L'| \to K(\mathbf{Z}_p,n)$, *there exists an extension of* $f \circ \psi_L|_{\psi_L^{-1}(|L'|)}$.

Sketch of Proof. We give its proof by using Edwards-Walsh's modification by Dranishnikov [6]. By the induction on $\dim L$, ψ_L is constructed to satisfy the following:

(1) $\psi_L^{-1}(L^{(n)}) = L^{(n)}$ is a subcomplex of $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$ and $\psi_L|_{|L^{(n)}|} = id_{|L^{(n)}|}$.

Let σ be a simplex of L with $\dim\sigma = n+1$. Let $K(\sigma)$ be an Eilenberg-MacLane space of type $(\mathbf{Z}_p, n)$ obtained from $\partial\sigma$ by attaching an (n+1)-cell by a map of degree p. Hence

(2) $K(\sigma)^{(n)} = \partial\sigma$ and $K(\sigma)^{(n+1)} = \partial\sigma \cup_\alpha B^{n+1}$, where $\alpha\colon \partial B^{n+1} \to \partial\sigma$ is a map of degree p.

If $\dim\sigma \geq n+2$ and $n \geq 2$, then $K(\sigma) = K_1(\sigma) \cup K_2(\sigma) \cup \ldots$ such that

(3) $K_1(\sigma) = \bigcup_{\tau \precneqq \sigma} K(\tau)$, where the union is taken over all proper faces τ of σ,
(4) for $i = 2,3,\ldots$, $K_i(\sigma)$ is obtained from $K_{i-1}(\sigma)$ by attaching to $K_{i-1}(\sigma)^{(n+i-1)}$ a finite collection of $(n+i)$-cells killing the $(n+i-1)$-th homotopy group.

If $\dim\sigma \geq n+2$ and $n = 1$, then $K(\sigma) = K_1(\sigma) \cup K_2(\sigma) \cup \ldots$ such that

(5) $K_1(\sigma)$ is obtained from $\bigcup_{\tau \precneqq \sigma} K(\tau)$, by attaching finite collection of 2-cells abelizing the fundamental group,
(6) for $i = 2,3,\ldots$, $K_i(\sigma)$ is obtained from $K_{i-1}(\sigma)$ by attaching to $K_{i-1}(\sigma)^{(n+i-1)}$ a finite collection of $(n+i)$-cells killing the $(n+i-1)$-th homotopy group.

Then we construct as

(7) $\psi_L^{-1}(\sigma)$ is the mapping cylinder M_σ of the embedding $j_\sigma\colon \psi_L^{-1}(\partial\sigma) \hookrightarrow K(\sigma)$,
(8) $\psi_L|M_\sigma$ is the cone of $\psi_L|_{\psi_L^{-1}(\partial\sigma)}$ such that $\psi_L(K(\sigma))$ is the barycentre of σ.

Hence for each simplex σ of $\dim\sigma \geq n+1$, we have the property:

(9) if $n \geq 2$,

$$\psi_L^{-1}(\sigma)^{(n+1)} = \sigma^{(n)} \times [0,1] \cup_{\alpha_1} B^{n+1} \cup_{\alpha_2} \cdots \cup_{\alpha_{r_\sigma}} B^{n+1},$$

where for each (n+1)-dimensional face τ_i of σ, $\alpha_i\colon \partial B^{n+1} \to \partial\tau_i \times \{1\}$ is a map of degree p,

(10) if $n = 1$,

$$\psi_L^{-1}(\sigma)^{(2)} = \sigma^{(1)} \times [0,1] \cup_{\alpha_1} B^2 \cup_{\alpha_2} \cdots \cup_{\alpha_{r_\sigma}} B^2 \cup_{\beta_1} B^2 \cup_{\beta_2} \cdots \cup_{\beta_{k_\sigma}} B^2,$$

where for each 2-dimensional face τ_i of σ, $\alpha_i\colon \partial B^2 \to \partial\tau_i \times \{1\}$ is a map of degree p and the collection $\{[\beta_1], \dots [\beta_{k_\sigma}]\}$ generates the commutator subgroup of $\pi_1\left(\sigma^{(1)} \times [0,1] \cup_{\alpha_1} B^2 \cup_{\alpha_2} \cdots \cup_{\alpha_{r_\sigma}} B^2\right)$. □

3. Characterizations for metrizable spaces

Let us establish definitions. Let K be a simplicial complex and $f, g\colon X \to |K|$ be maps. We say that g is a *K-modification* of f if for each $x \in X$ and $\sigma \in K$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. Let $\mathcal{U}$ be an open cover of X. Then a map $b\colon X \to |\mathcal{N}(\mathcal{U})|$ is called *$\mathcal{U}$-normal* map if $b^{-1}(\mathrm{st}(U,\mathcal{U})) = U$ for each $U \in \mathcal{U}$ and b is essential on each simplex of $\mathcal{N}(\mathcal{U})$ (i.e. $b|_{b^{-1}(\sigma)}\colon b^{-1}(\sigma) \to \sigma$ is a essential map for each $\sigma \in \mathcal{N}(\mathcal{U})$). Note that if $\mathcal{U}$ is a locally finite, then $\mathcal{U}$-normal map exists.

3.1. Definition. Let Q, P be polyhedra, G be an abelian group, $\mathcal{U}$ be an open cover of P and n be a natural number. We say that a map $\psi\colon Q \to P$ is *$(G,n,\mathcal{U})$-approximable* if there exists a triangulation L of P such that for any triangulation M of Q there is a PL-map $\psi'\colon |M^{(n)}| \to |L^{(n)}|$ satisfying the following conditions:

(i) $\left(\psi', \psi|_{|M^{(n)}|}\right) \leq \mathcal{U}$,

(ii) for any map $\alpha\colon |L^{(n)}| \to K(G,n)$, there exists an extension $\beta\colon |M^{(n+1)}| \to K(G,n)$ of $\alpha \circ \psi'$.

3.2. Definition. Let G be an abelian group and n be a natural number. A map $f\colon X \to P$ of a metrizable space X to a polyhedron P is called *(G,n)-cohomological* if for any open cover $\mathcal{U}$ of P there exist a polyhedron Q and maps $\varphi\colon X \to Q$, $\psi\colon Q \to P$ such that

(i) $(\psi \circ \varphi, f) \leq \mathcal{U}$,

(ii) ψ is $(G,n,\mathcal{U})$-approximable.

3.3. Theorem. *Let X be a metrizable space, p be a prime number and n be a natural number. Then X has cohomological dimension with respect to $\mathbf{Z}_p$ of less than and equal to n if and only if every map f of X to a polyhedron P is $(\mathbf{Z}_p, n)$-cohomological.*

Proof of necessity. Suppose that $c\text{-}\dim_{\mathbf{Z}_p} X \leq n$. Let $f\colon X \to P$ be a map of X to a polyhedron P and $\mathcal{U}$ be an open cover of P. Then take a star refinement $\mathcal{U}_0$ of $\mathcal{U}$.

First, we show that there exist a simplicial complex K and maps $\varphi\colon X \to |K|$, $\psi\colon |K| \to P$ such that

(1) if $\sigma \in K$, there exists $U \in \mathcal{U}_0$ with $\psi(\sigma) \subseteq U$,

(2) for each $x \in X$ if $\varphi(x) \in \operatorname{Int}\sigma$, $\sigma \in K$, there exists $U \in \mathcal{U}_0$ with $\psi(\sigma) \cup \{f(x)\} \subseteq U$,

(3) there exist a triangulation L of P and a PL-map $\psi'\colon |K^{(n)}| \to |L^{(n)}|$ such that

 (i) $(\psi', \psi|_{|K^{(n)}|}) \leq \mathcal{U}_0$

 (ii) for any map $\alpha\colon |L^{(n)}| \to K(G,n)$ there is an extension $\beta\colon |K^{(n+1)}| \to K(G,n)$ of $\alpha \circ \psi'$.

By J. H. C. Whitehead's theorem [25], take a triangulation L of P such that

(4) $\operatorname{st}\left\{\bar{\operatorname{st}}(v, L) : v \in L^{(0)}\right\} \prec \mathcal{U}_0$.

We will construct a map $c\colon X \to \mathrm{EW}_{\mathbf{Z}_p}(L,n)$ such that

(5) $c|_{f^{-1}(|L^{(n)}|)} = f|_{f^{-1}(|L^{(n)}|)}$,

(6) $c\left(f^{-1}(\sigma)\right) \subseteq \psi_L^{-1}(\sigma)$ for $\sigma \in L$, where $\psi_L\colon \mathrm{EW}_{\mathbf{Z}_p}(L,n) \to L$ is the map constructed in Lemma 2.1.

We define the map $c_n \equiv f|_{f^{-1}(|L^{(n)}|)}\colon f^{-1}\left(|L^{(n)}|\right) \to |L^{(n)}| \subseteq \mathrm{EW}_{\mathbf{Z}_p}(L,n)$. Inductively, suppose that for $n \leq k$ we have defined the function $c_k\colon f^{-1}\left(|L^{(k)}|\right) \to \mathrm{EW}_{\mathbf{Z}_P}(L,n)$ such that $c_k|_{f^{-1}(\sigma)}\colon f^{-1}(\sigma) \to \psi_L^{-1}(\sigma) \subseteq \mathrm{EW}_{\mathbf{Z}_p}(L,n)$ is continuous and $c_k|_{f^{-1}(\sigma)} = c_k|_{f^{-1}(\tau)}$ on $f^{-1}(\sigma) \cap f^{-1}(\tau)$ for $\sigma, \tau \in L^{(k)}$. Now, let $\sigma \in L$ with $\dim\sigma = k+1$. By the construction of c_k and $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$, $c_k|_{f^{-1}(\partial\sigma)}\colon \partial\sigma \to \psi_L^{-1}(\sigma)$ is continuous. Hence by $c\text{-}\dim_{\mathbf{Z}_p} f^{-1}(\sigma) \leq c\text{-}\dim_{\mathbf{Z}_p} X \leq n$ and (i) in Lemma 2.1, we have an continuous extension $c_\sigma\colon f^{-1}(\sigma) \to \psi_L^{-1}(\sigma)$ of $c_k|_{f^{-1}(\partial\sigma)}$. Define c_{k+1} to be c_σ on $f^{-1}(\sigma)$ for $\sigma \in L$ with $\dim\sigma = k+1$. Finally, we define c to be $\bigcup_{k=n}^{\infty} c_k$. Then since X is compactly generated, the function c is continuous.

We define an open cover $\mathcal{B} = \{B_\sigma : \sigma \in L\}$ in the following way:

$$B_\sigma \equiv \mathrm{EW}_{\mathbf{Z}_p}(L,n) \setminus \bigcup\{\psi_L^{-1}(\tau) : \sigma \cap \tau = \emptyset\}.$$

Then note that we have

(7) $\psi_L^{-1}(\sigma) \subseteq B_\sigma$

(8) if $x \in B_\sigma$ and $x \in \psi_L^{-1}(\tau)$, $\sigma \cap \tau \neq \emptyset$.

Since $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$ is LC^n, for a star refinement $\mathcal{B}_1$ of $\mathcal{B}$, there exists an open refinement $\mathcal{B}_2$ of $\mathcal{B}_1$ such that if K is a simplicial complex of $\dim K \leq n+1$, then every partial realization of K in $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$ relative to $\mathcal{B}_2$ extended to a full realization relative to $\mathcal{B}_1$ [2]. Select a star refinement $\mathcal{B}_3$ of $\mathcal{B}_2$.

Then by [21, Lemma 9.6], there exist an open cover $\mathcal{V}$ of X refining $f^{-1}(\mathcal{U}_0) \wedge c^{-1}(\mathcal{B}_3)$ and maps $\varphi\colon X \to |\mathcal{N}(\mathcal{V})|$, $\psi\colon |\mathcal{N}(\mathcal{V})| \to P$ such that

(9) φ is $\mathcal{V}$-normal,

(10) $\psi \circ \varphi$ is L-modification of f,

(11) if $\sigma \in \mathcal{N}(\mathcal{V})$, there exists $U \in \mathcal{U}_0$ with $f\left(\varphi^{-1}(\sigma)\right) \cup \psi(\sigma) \subseteq U$.

Then these $\mathcal{N}(\mathcal{V})$, φ and ψ satisfy the conditions (1)-(3).

It is easily seen that (11) implies (1) and (2). It remain to prove that (3) holds.

We shall construct a map $\psi_0\colon |\mathcal{N}(\mathcal{V})^{(n+1)}| \to \mathrm{EW}_{\mathbf{Z}_p}(L,n)$ in the following way: note that if $\langle U\rangle \in \mathcal{N}(\mathcal{V})^{(n+1)}$, there exists $B_U \in \mathcal{B}_3$ with $U \subseteq c^{-1}(B_U)$. ψ_0 on $|\mathcal{N}(\mathcal{V})^{(0)}|$ is defined by an element $\psi_0(\langle U\rangle) \in B_U$ for each $\langle U\rangle \in \mathcal{N}(\mathcal{V})^{(0)}$. Let $\langle U_0,\dots,U_m\rangle \in \mathcal{N}(\mathcal{V})^{(n+1)}$. Then by $\emptyset \neq U_0\cap\cdots\cap U_m \subseteq c^{-1}(B_{U_0})\cap\cdots\cap c^{-1}(B_{U_m})$, we have

$$\psi_0(\{\langle U_0\rangle,\dots,\langle U_m\rangle\}) \subseteq \mathrm{st}(B_{U_0},\mathcal{B}_3) \subseteq B \text{ for some } B \in \mathcal{B}_2.$$

It show that ψ_0 is a partial realization of $\mathcal{N}(\mathcal{V})^{(n+1)}$ in $\mathrm{EW}_{\mathbf{Z}_p}(L,n)$ relative to $\mathcal{B}_2$. Therefore, by the construction of $\mathcal{B}_2$, we may define ψ_0 to be a *full realization relative to* $\mathcal{B}_1$. Then by the same way in [21, p245 (8)] we can show that

(12) if $t \in |\mathcal{N}(\mathcal{V})^{(n+1)}|$ with $\psi(t) \in \mathrm{Int}\,\delta$ and $\psi_0(t) \in \psi_L^{-1}(\tau)$ for $\delta,\tau \in L$, then there exist $\sigma,\lambda \in L$ such that $\delta \prec \sigma$ and $\sigma\cap\lambda \neq \emptyset \neq \lambda\cap\tau$.

Now, by the property (v) in Lemma 2.1, we can choose

(13) a cellular map $\psi_1\colon |\mathcal{N}(\mathcal{V})^{(n+1)}| \to \mathrm{EW}_{\mathbf{Z}_p}(L,n)^{(n+1)}$ such that for each $t \in |\mathcal{N}(\mathcal{V})^{(n+1)}|$, if $\psi_0(t) \in \psi_L^{-1}(\tau)$, then $\psi_1(t) \in \psi_L^{-1}(\tau)^{(n+1)}$.

By the simplicial approximation theorem, we assume that ψ_1 is PL.

If $n \geq 2$, by the properties (9) and (1) in Lemma 2.1, we have

$$\mathrm{EW}_{\mathbf{Z}_p}(L,n)^{(n+1)} = |L^{(n)}| \cup \bigcup\{\partial\sigma \times [0,1] \cup_{\alpha_\sigma} B_\sigma^{n+1} : \sigma \in L, \dim\sigma = n+1\},$$

where $\alpha_\sigma\colon \partial B_\sigma^{n+1} \to \partial\sigma$ is a map of degree p. For each (n+1)-simplex σ of L, choose a point $z_\sigma \in B_\sigma^{n+1} \setminus \partial B_\sigma^{n+1}$, and take the retraction

$$r\colon \mathrm{EW}_{\mathbf{Z}_p}(L,n)^{(n+1)} \setminus \{z_\sigma : \sigma \in L, \dim\sigma = n+1\} \to |L^{(n)}|$$

induced by the compositions of the radial projection of $B_\sigma^{n+1} \setminus \{z_\sigma\}$ onto $\partial\sigma \times \{1\}$ and the natural projection of $\partial\sigma \times [0,1]$ onto $\partial\sigma \times \{0\} \subseteq |L^{(n)}|$.

If $n = 1$, for every simplex σ of $\dim\sigma \geq 2$, $\psi_L^{-1}\left(\sigma^{(2)}\right)$ may be represented as the form (10) in Lemma 2.1:

$$\psi_L^{-1}(\sigma)^{(2)} = \sigma^{(1)} \times [0,1] \cup_{\alpha_1} B^2 \cup_{\alpha_2} \cdots \cup_{\alpha_{r_\sigma}} B^2 \cup_{\beta_1} B^2 \cup_{\beta_2} \cdots \cup_{\beta_{k_\sigma}} B^2.$$

Then choose points $u_1^\sigma,\dots,u_{r_\sigma}^\sigma, v_1^\sigma,\dots,v_{k_\sigma}^\sigma$ of $\psi_L^{-1}(\sigma^{(1)})^{(2)} \setminus \sigma^{(1)} \times [0,1]$ for each B^2 and the retraction $r\colon \mathrm{EW}_{\mathbf{Z}_p}(L,n)^{(2)} \setminus \{u_1^\sigma,\dots,u_{r_\sigma}^\sigma, v_1^\sigma,\dots,v_{k_\sigma}^\sigma : \sigma \in L, \dim\sigma \geq 2\} \to |L^{(1)}|$ induced by the compositions of the radial projections of $B^2 \setminus \{u_i^\sigma\}$ or $B^2 \setminus \{v_j^\sigma\}$ onto S^1 and the natural projection of $\sigma^{(1)} \times [0,1]$ onto $\sigma^{(1)} \times \{0\} \subseteq |L^{(1)}|$.

In both cases, we put

$$\psi' \equiv r \circ \psi_1|_{|\mathcal{N}(\mathcal{V})^{(n)}|}\colon |\mathcal{N}(\mathcal{V})^{(n)}| \to |L^{(n)}|.$$

Then the map ψ' holds the conditions (i),(ii). First, we show the condition (i). Let $t \in |\mathcal{N}(\mathcal{V})^{(n)}|$. By (12), there exist $\sigma, \lambda, \tau \in L$ such that $\sigma \cap \lambda \neq \emptyset \neq \lambda \cap \tau$ and $\psi(t) \in \sigma$, $\psi_0(t) \in \psi_L^{-1}(\tau)$. Then since $\psi_1(t)$ is an element of $\psi_L^{-1}(\tau)^{(n)}$, we have $\psi'(t) \in \tau$. Hence, we have $\psi(t), \psi'(t) \in \bar{\mathrm{st}}(\lambda, L) \subseteq U$ for some $U \in \mathcal{U}_0$ (see (4)). Next, we must show the condition (ii). But, it is easy to show that. Hence, we omitted it here.

Now, we shall show that f is $(\mathbf{Z}_p, n)$-cohomological . By (2), we can easily see that $(\psi \circ \varphi, f) \leq \mathcal{U}$. So, we show that ψ is $(\mathbf{Z}_p, n, \mathcal{U})$-approximable .

Let M be a triangulation of $|K|$. Note that for a simplicial approximation j of $id_{|M|}\colon |M| = |K| \to |K|$ with respect to K, we have that

$$j\left(|M^{(n+1)}|\right) \subseteq |K^{(n+1)}| \text{ and } j\left(|M^{(n)}|\right) \subseteq |K^{(n)}|.$$

Then by (1) and (3), we can easily see that the map

$$\psi'' \equiv \psi' \circ j\colon |M^{(n)}| \to |L^{(n)}|$$

holds the conditions. □

The reverse implication is proved by the standard way [21]. First, we need some notations.

We may assume that the Eilenberg-MacLane space $K(\mathbf{Z}_p, n)$ is a metrizable, locally compact separable space. Then by the Kuratowski-Wojdyslawski's theorem, we can consider that $K(\mathbf{Z}_p, n)$ is a closed subset of a convex subset C of a normed linear space E. Note that C is AR(metrizable spaces). Since $K(\mathbf{Z}_p, n)$ is ANR, there exist a closed neighborhood F in C and a retraction $r\colon F \to K(\mathbf{Z}_p, n)$. Further, we can choose an open cover $\mathcal{W}_0$ of $\mathrm{Int}_C F$ such that

(1) for any space Z and any maps $\alpha, \beta\colon Z \to F$ with $(\alpha, \beta) \leq \mathcal{W}_0$, the maps $r \circ \alpha, r \circ \beta\colon Z \to K(\mathbf{Z}_p, n)$ are homotopic in $K(\mathbf{Z}_p, n)$.

Then we take an open, *convex* cover $\mathcal{W}$ of C such that

(2) if $W \in \mathcal{W}$ with $W \cap K(\mathbf{Z}_p, n) \neq \emptyset$, there exists $U \in \mathcal{W}_0$ with $\mathrm{st}(W, \mathcal{W}) \subseteq U$.

Select a star refinement $\mathcal{V}$ of $\mathcal{W}$.

Let $h_0\colon C \to |\mathcal{N}(\mathcal{V})|$ be a Kuratowski's map with respect to $\mathcal{V}$ and define a map $h_1\colon |\mathcal{N}(\mathcal{V})| \to C$ in the following way: a map h_1 on $|\mathcal{N}(\mathcal{V})^{(0)}|$ is defined by an element $h_1(\langle V \rangle) \in V$ for each $\langle V \rangle \in |\mathcal{N}(\mathcal{V})^{(0)}|$. Next, by using the convexity of C, we extend h_1 *linearly* on each simplex $|\mathcal{N}(\mathcal{V})|$. Let $\sigma = \langle V_0, \dots, V_m \rangle \in |\mathcal{N}(\mathcal{V})|$. Then by $V_0 \cap \cdots \cap V_m \neq \emptyset$,

$$h_1\left(\{\langle V_0 \rangle, \dots, \langle V_m \rangle\}\right) \subseteq \mathrm{st}(V_o, \mathcal{V}) \subseteq W_\sigma \text{ for some } W_\sigma \in \mathcal{W}.$$

Thus, by the construction of h_1, we have $h_1(\sigma) \subseteq W_\sigma$.

Let $\mathcal{N}_1$ be a subcomplex $\mathcal{N}(\{V \in \mathcal{V} : V \cap K(\mathbf{Z}_p, n) \neq \emptyset\})$ of $\mathcal{N}(\mathcal{V})$. Let $\mathcal{N}_0$ be a simplicial neighborhood of $\mathcal{N}_1$ in $\mathcal{N}(\mathcal{V})$ such that if $\langle V_0 \rangle \in \mathcal{N}_0$, there exists $\langle V_1 \rangle \in \mathcal{N}_1$ with $V_0 \cap V_1 \neq \emptyset$. Then we can easily see the followings:

(3) for each $x \in K(\mathbf{Z}_p, n)$, there exists $W \in \mathcal{W}$ with $x, h_1 \circ h_0(x) \in W$,

(4) $h_1(|\mathcal{N}_0|) \subseteq \mathrm{st}(K(\mathbf{Z}_p, n), \mathcal{W}) \subseteq F$,

(5) $h_0(K(\mathbf{Z}_p, n)) \subseteq |\mathcal{N}_1| \subseteq |\mathcal{N}_0|$.

Proof of sufficiency. Let A be a closed subset of X and $h\colon A \to K(\mathbf{Z}_p, n)$ be a map. We consider the above-mentioned nerve $\mathcal{N}(\mathcal{V})$ and maps h_0, h_1. We take an open cover $\mathcal{U}$ of $|\mathcal{N}(\mathcal{V})|$ such that

(6) $\mathrm{st}^3(|\mathcal{N}_1|, \mathcal{U}) \subseteq |\mathcal{N}_0|$,

(7) $\mathrm{st}^3(\mathcal{U}) \prec h_1^{-1}(\mathcal{W})$,

and choose a subdivision $\mathcal{N}$ of $\mathcal{N}(\mathcal{V})$ such that if $\sigma \in \mathcal{N}$ there exists $U \in \mathcal{U}$ with $\sigma \subseteq U$.

Since C is AE, there is an extension $H\colon X \to C$ of h. Then by the assumption, the map $h_0 \circ H\colon X \to |\mathcal{N}(\mathcal{V})|$ is $(\mathbf{Z}_p, n)$-cohomological. Hence, there exist a polyhedron Q and maps $\varphi\colon X \to Q$, $\psi\colon Q \to |\mathcal{N}(\mathcal{V})|$ such that

(8) $(\psi \circ \varphi, h_0 \circ H) \leq \mathcal{U}$,

(9) ψ is $(\mathbf{Z}_p, n, \mathcal{U})$-approximable.

By using the simplicial approximation theorem, we obtain a triangulation M of Q and a simplicial approximation $\psi^*\colon M \to \mathcal{N}$ of ψ. Then by (8),(9), we have

(10) $(\psi^* \circ \varphi, h_0 \circ H) \leq \mathrm{st}\,\mathcal{U}$,

(11) ψ^* is $(\mathbf{Z}_p, n, \mathrm{st}\,\mathcal{U})$-approximable.

Now, by (11) with respect to M, there exist a triangulation L and a PL-map $\psi'\colon |M^{(n)}| \to |L^{(n)}|$ such that

(12) $(\psi', \psi^*|_{|M^{(n)}|}) \leq \mathrm{st}\,\mathcal{U}$,

(13) for any map $\alpha\colon |L^{(n)}| \to K(\mathbf{Z}_p, n)$, there exists an extension $\beta\colon |M^{(n+1)}| \to K(\mathbf{Z}_p, n)$ of $\alpha \circ \psi'$.

Claim. *There exists a map $\xi\colon Q \to K(\mathbf{Z}_p, n)$ such that $\xi|_{\psi^{*-1}(|\mathcal{N}_0|)} = r \circ h_1 \circ \psi^*|_{\psi^{*-1}(|\mathcal{N}_0|}$*

Construction of ξ. First, we shall see that

(14) for each $x \in D \equiv \psi^{*-1}(|\mathcal{N}_0|) \cap |M^{(n)}|$, there exists $U \in \mathcal{W}_0$ such that $h_1 \circ \psi^*(x), h_1 \circ \psi'(x) \in U$.

By (12), there exist $U_1, U_2, U_3 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \emptyset \neq U_2 \cap U_3$ and $\psi^*(x) \in U_1$, $\psi'(x) \in U_3$. Then by (7), we have $W \in \mathcal{W}$ with $h_1(U_1 \cup U_2 \cap U_3) \subseteq W$. Since $\psi^*(x) \in |\mathcal{N}_0|$, by (4), there exists $W' \in \mathcal{W}$ such that $h_1 \circ \psi^*(x) \in W$ and $W' \cap K(\mathbf{Z}_p, n) \neq \emptyset$. Hence by (2), we obtain $U \in \mathcal{W}_0$ such that $h_1 \circ \psi^*(x), h_1 \circ \psi'(x) \in \mathrm{st}(W', \mathcal{W}) \subseteq U$.

Therefore by (14) and (1), we see the followings:

(15) $h_1 \circ \psi'(D) \subseteq F$,

(16) $r \circ h_1 \circ \psi^*|_D \simeq r \circ h_1 \circ \psi'|_D$ in $K(\mathbf{Z}_p, n)$.

Since D is a subpolyhedron of $|M^{(n)}|$ and ψ' is PL, $\psi'(D)$ is subpolyhedron of $|L^{(n)}|$. Hence, from $\pi_q(K(\mathbf{Z}_p, n)) = 0$ for $q < n$ (if $n = 1$, the path-connectedness of $K(\mathbf{Z}_p, n)$), there exists an extension

$$\alpha \colon |L^{(n)}| \to K(\mathbf{Z}_p, n)$$

of $r \circ h_1|_{\psi'(D)} \colon \psi'(D) \to K(\mathbf{Z}_p, n)$.

Then by (13), we have an extension

$$\beta \colon |M^{(n+1)}| \to K(\mathbf{Z}_p, n)$$

of $\alpha \circ \psi'$.

Now, put

$$R \equiv |M^{(n+1)}| \setminus \bigcup \{\operatorname{Int} \sigma : \sigma \in M, \dim \sigma = n+1, \sigma \subseteq \psi^{*-1}(|\mathcal{N}_0|)\}.$$

Then since for each $x \in D \subseteq R$ we have $\beta(x) = \alpha \circ \psi'(x) = r \circ h_1 \circ \psi'(x)$,

$$\beta|_D \simeq r \circ h_1 \circ \psi'(x)|_D \simeq r \circ h_1 \circ \psi^*|_D \quad \text{in } K(\mathbf{Z}_p, n). \tag{17}$$

By the homotopy extension theorem, there exists an extension $\xi_R \colon R \to K(\mathbf{Z}_p, n)$ of $r \circ h_1 \circ \psi^*|_D$.

Since for $\sigma \in M$ with $\dim \sigma = n+1$ and $\sigma \subseteq \psi^{*-1}(|\mathcal{N}_0|)$, we have $\xi_R|_{\partial\sigma} = r \circ h_1 \circ \psi^*|_{\partial\sigma}$, there exists an extension $\xi_{n+1} \colon |M^{(n+1)}| \to K(\mathbf{Z}_p, n)$ of ξ_R such that $\xi_{n+1}|_{\psi^{*-1}(|\mathcal{N}_0|) \cap |M^{(n+1)}|} = r \circ h_1 \circ \psi^*|_{\psi^{*-1}(|\mathcal{N}_0|) \cap |M^{(n+1)}|}$.

Hence, we can define a map $\xi' \colon \psi^{*-1}(|\mathcal{N}_0|) \cup |M^{(n+1)}| \to K(\mathbf{Z}_p, n)$ by the following:

$$\xi' \equiv \left(r \circ h_1 \circ \psi^*|_{\psi^{*-1}(|\mathcal{N}_0|)}\right) \cup \xi_{n+1}.$$

Therefore from $\pi_q(K(\mathbf{Z}_p, n)) = 0$ for $q > n$, we obtain an extension $\xi \colon Q \to K(\mathbf{Z}_p, n)$ of ξ' such that $\xi|_{\psi^{*-1}(|\mathcal{N}_0|)} = r \circ h_1 \circ \psi^*|_{\psi^{*-1}(|\mathcal{N}_0|)}$. It completes the construction.

Now, we put

$$h' \equiv \xi \circ \varphi \colon X \to K(\mathbf{Z}_p, n).$$

Then to complete the proof it suffices to prove

$$h'|_A \simeq h \text{ in } K(\mathbf{Z}_p, n). \tag{18}$$

First, we shall see that

$$\psi^* \circ \varphi(A) \subseteq |\mathcal{N}_0|.$$

Let $a \in A$. By (10), there exist $U_1, U_2, U_3 \in \mathcal{U}$ such that

(19) $U_1 \cap U_2 \neq \emptyset \neq U_2 \cap U_3$ and $\psi^* \circ \varphi(a) \in U_1$, $h_0 \circ H(a) \in U_3$.

Then since $h_0 \circ H(a) = h_0 \circ h(a) \in h_0(K(\mathbf{Z}_p, n)) \subseteq |\mathcal{N}_1|$, we have $\psi^* \circ \varphi(a) \in |\mathcal{N}_0|$ by (6).

Hence, by Claim, we have for each $a \in A$ $h'(a) = \xi \circ \varphi(a) = r \circ h_1 \circ \psi^* \circ \varphi(a)$. Therefore, by (1), it suffices to see that

(20) there exists $U \in \mathcal{W}_0$ such that $h_1 \circ \psi^* \circ \varphi(a), h(a) \in U$.

Let $U_1, U_2, U_3 \in \mathcal{U}$ with the property (19). By (7), there exists $W \in \mathcal{W}$ such that $U_1 \cup U_2 \cup U_3 \subseteq h_1^{-1}(W)$. By (3) we choose $W' \in \mathcal{W}$ such that $h(a), h_1 \circ h_0 \circ h(a) \in W'$. Therefore, since $h(a) \in K(\mathbf{Z}_p, n)$, there exists $U \in \mathcal{W}_0$ such that

$$h_1 \circ \psi^* \circ \varphi(a), h(a) \in \mathrm{st}(W', \mathcal{W}) \subseteq U.$$

It completes the proof. □

4. Approximable dimension

4.1. Definition. A space X has *approximable dimension with respect to a coefficient group G of less than and equal to n (abbreviated, a-$\dim_G X \leq n$)* provided that for every polyhedron P, map $f\colon X \to P$ and open cover $\mathcal{U}$, there exist a polyhedron Q and maps $\varphi\colon X \to Q$, $\psi\colon Q \to P$ such that

(i) $(\psi \circ \varphi, f) \leq \mathcal{U}$,
(ii) ψ is $(G, n, \mathcal{U})$-approximable.

First, we state fundamental inequalities of a-$\dim_G$.

4.2. Theorem. *For a metrizable space X and an arbitrary abelian group G, we hold the following inequalities:*

$$c\text{-}\dim_G X \leq a\text{-}\dim_G X \leq \dim X.$$

Proof. The second inequality is trivial. We can see the first inequality by the strategy similar to the proof of the sufficiency in Theorem 3.3. □

As we will show in latter sections, our approach of a-$\dim_G$ gives useful applications. In general, a-$\dim_G$ is different from c-$\dim_G$. However, in special cases of coefficient group G, a-$\dim_G$ coincides with c-$\dim_G$.

4.3. Theorem. *If $G = \mathbf{Z}$ or $\mathbf{Z}_p$, where p is a prime number, for every metrizable space X, we have*

$$a\text{-}\dim_G X = c\text{-}\dim_G X.$$

Proof. From Theorem 3.3, 4.2, we see the fact. □

5. Resolutions for metrizable spaces

By a polyhedron we mean the space $|K|$ of a simplicial complex K with the *Whitehead topology* (denoted by $|K|_w$). We may define a topology for $|K|$ by means of a uniformity in [Appendix, 22] (denoted by $|K|_u$).

5.1. Theorem. *Let X be a metrizable space having approximable dimension with respect to an abelian group G of less than and equal to n. Then there exist an n-dimensional metrizable space Z and a perfect UV^{n-1}-surjection $\pi\colon Z \to X$ such that for $x \in X$, the set $[\pi^{-1}(x), K(G,n)]$ of homotopy classes is trivial.*

Proof. The strategy is like the construction of Walsh-Rubin [24,22].

Let d be a metric for X and let $\{\mathcal{U}_i : i \in \mathrm{N} \cup \{0\}\}$ be a sequence of open covers of X where each $\mathcal{U}_i$ consists of all $1/(i+1)$-neighborhoods.

First, we shall construct the followings:

open covers $\mathcal{V}_i$ of X whose nerves $\mathcal{N}(\mathcal{V}_i)$ are locally finite dimensional, maps $b_i\colon X \to |\mathcal{N}(\mathcal{V}_i)|$ for $i \geq 0$, $f_i^*, f_i\colon |\mathcal{N}(\mathcal{V}_i)| \to |\mathcal{N}(\mathcal{V}_{i-1})|$ for $i \geq 1$ and sequences $\mathcal{N}_i^j, j \in \mathrm{N} \cup \{0\}$ of subdivisions of $\mathcal{N}(\mathcal{V}_i)$ for $i \geq 0$ such that

(1) $\bar{\mathcal{S}}_i^{j+1} \prec^* \mathcal{S}_i^j$ for $j \geq 0$,
(2) b_i is normal with respect to $b_i^{-1}(\mathcal{S}_i^j)$ and $\mathcal{N}_i^j$ for $j \geq 0$,
(3) $f_i\colon \mathcal{N}_i^0 \to \mathcal{N}_{i-1}^3$ is simplicial for $i \geq 1$,
(4) $f_i \circ b_i$ is $\mathcal{N}_{i-1}^j$-modification of b_{i-1}, $0 \leq j \leq 3$ for $i \geq 1$,
(5) f_i maps each compact set in $|\mathcal{N}_i|_u$ onto a compact set in $|\mathcal{N}_{i-1}|_u$ which is contained in a finite union of simplexes of $\mathcal{N}_{i-1}$,
(6) $\mathcal{S}_i^0 \prec f_i^{-1}(\mathcal{S}_{i-1}^3)$ for $i \geq 1$,
(7) $\bar{\mathcal{S}}_i^k \prec f_i^{-1}(\mathcal{S}_{i-1}^{k+3})$ for $k \geq 1$ and $\bar{\mathcal{S}}_i^k \prec f_i^{*-1}(\mathcal{S}_{i-1}^{k+3})$ for $k \geq 4$,
(8) $\mathcal{V}_i \prec \mathcal{U}_i \wedge b_{i-1}^{-1}(\mathcal{S}_{i-1}^3) \wedge b_{i-2}^{-1}(\mathcal{S}_{i-2}^6) \wedge \cdots \wedge b_0^{-1}(\mathcal{S}_0^{3i})$,

where we regard $|\mathcal{N}_i|_u$ as the uniform space with the uniform topology induced by the uniform base $\{\mathcal{S}_i^j\}_{j=0}^\infty$.

Further, we shall construct continuous (w.r.t. the Whitehead topology), uniformly continuous (w.r.t. the uniform topology) PL-maps $g_i\colon |(\mathcal{N}_i^3)^{(n)}| \to |(\mathcal{N}_{i-1}^3)^{(n)}|$ such that

(9) for each $t \in |(\mathcal{N}_i^3)^{(n)}|$, there exist $\sigma, \tau \in \mathcal{N}_{i-1}^2$ such that $f_i(t) \in \sigma$, $g_i(t) \in \tau$ and $\sigma \cap \tau \neq \emptyset$,
(10) for any map $\alpha\colon |(\mathcal{N}_{i-1}^3)^{(n)}|_w \to K(G,n)$, there exists an extension $\beta\colon |(\mathcal{N}_i^3)^{(n+1)}| \to K(G,n)$ of $\alpha \circ g_i\colon |(\mathcal{N}_i^3)^{(n)}|_w \to |(\mathcal{N}_{i-1}^3)^{(n)}|_w \to K(G,n)$,
(11) for each $x \in |\mathcal{N}_i|$, $g_i\left(\mathrm{st}(x, \bar{\mathcal{S}}_i^2) \cap |(\mathcal{N}_i^3)^{(n)}|\right)$ is a Whitehead (i.e. finite) compact polyhedral subset of $|\mathcal{N}_{i-1}|$.

Let us start the construction. We take an open refinement $\mathcal{V}_0$ of $\mathcal{U}_0$ in X whose nerve $\mathcal{N}(\mathcal{V}_0)$ is locally finite dimensional and $\mathcal{V}_0$-normal map $b_0\colon X \to |\mathcal{N}(\mathcal{V}_0)|$. We define $\mathcal{N}_0^j$ to be a subdivision of $\mathrm{Sd}_{2j}\,\mathcal{N}(\mathcal{V}_0)$ for $j = 0,1,2$ with $\bar{\mathcal{S}}_0^j \prec \mathcal{S}_0^{j-1}$. By using [22, Proposition A.3], for the cover $\mathcal{E}_0 \equiv \left\{\mathrm{st}(x, \bar{\mathcal{S}}_0^2) : x \in |\mathcal{N}(\mathcal{V}_0)|\right\}$, we obtain an open cover $\mathcal{B}_0$ of $|\mathcal{N}(\mathcal{V}_0)|$ and a PL, $\mathcal{N}_0^2$-modification $r_0\colon |\mathcal{N}_0^2| \to |\mathcal{N}_0^2|$ of the identity such that

$(12)_0$ $r_0(\mathrm{Cl}\,B)$ is compact for $B \in \mathcal{B}_0$,

$(13)_0$ $\mathrm{Cl}\,B \cup r_0(\mathrm{Cl}\,B) \subseteq E$ for some $E \in \mathcal{E}_0$.

Since b_0 is (G,n)-cohomological, from the similar argument to the proof of the necessity in Theorem 3.3 we can take the followings:

subdivision $\mathcal{N}_0^3$ of $\mathrm{Sd}_2\,\mathcal{N}_0^2$, locally finite open cover $\mathcal{V}_1$ of X and maps $b_1\colon X \to |\mathcal{N}(\mathcal{V}_1)|$, $f_1^*\colon |\mathcal{N}(\mathcal{V}_1)| \to |\mathcal{N}_0^3|$ such that

$(14)_1$ $\bar{\mathcal{S}}_0^3 \prec^* \mathcal{S}_0^2 \wedge \mathcal{B}_0$,

$(15)_1$ $\mathcal{V}_1 \prec^* \mathcal{U}_1 \wedge b_0^{-1}(\mathcal{S}_0^3)$,

$(16)_1$ b_1 is $\mathcal{V}_1$-normal,

$(17)_1$ $f_1^* \circ b_1$ is $\mathcal{N}_0^3$-modification of b_0,

$(18)_1$ for each $\sigma \in \mathcal{N}(\mathcal{V}_1)$, there exists $U \in \mathrm{st}\,\mathcal{S}_0^3$ such that $b_0\left(b_1^{-1}(\sigma)\right) \cup f_1^*(\sigma) \subseteq U$,

$(19)_1$ for any triangulation M of $|\mathcal{N}(\mathcal{V}_1)|$, there exists a PL-map $p'\colon |M^{(n)}| \to |(\mathcal{N}_0^3)^{(n)}|$ such that

(i) $(p', f_1^*|_{|M^{(n)}|}) \le \{\bar{\mathrm{st}}(\lambda, \mathcal{N}_0^3) : \lambda \in \mathcal{N}_0^3\}$,

(ii) for any map $\alpha\colon |(\mathcal{N}_0^3)^{(n)}| \to K(G,n)$, there exists an extension $\beta\colon |M^{(n+1)}| \to K(G,n)$ of $\alpha \circ p'$.

Let $\mathcal{N}_0^{j+1}$ denote a subdivision of $\mathrm{Sd}_2\,\mathcal{N}_0^j$ with $\bar{\mathcal{S}}_0^{j+1} \prec^* \mathcal{S}_0^j$ for $j \ge 3$.

Now, let $|\mathcal{N}_0^3|_m$ denote $|\mathcal{N}_0^3|$ with the metric topology [19, p301]. Then there is a $\mathcal{N}_0^3$-modification $j_0\colon |\mathcal{N}_0^3|_m \to |\mathcal{N}_0^3|_w$ of the identity *function* [19, p302]. By the simplicial approximation theorem, we obtain a subdivision $\mathcal{N}_1$ of $\mathcal{N}(\mathcal{V}_1)$ and a simplicial approximation $f_1\colon \mathcal{N}_1 \to \mathcal{N}_0^3$ of $j_0 \circ f_1^*$. Let $\mathcal{N}_1^0$ denote $\mathcal{N}_1$. Then by the simpliciality of f_1 and $(17)_1$, we have

(20) $\mathcal{S}_1^0 \prec f_1^{-1}(\mathcal{S}_0^3)$,

(21) $f_1 \circ b_1$ is $\mathcal{N}_0^3$-modification of b_0.

We take a subdivisions $\mathcal{N}_1^{j+1}$ of $\mathcal{N}_1^0$ for $j = 0,1$ such that

(22) $\bar{\mathcal{S}}_1^{j+1} \prec^* \mathcal{S}_1^j$ for $j = 0,1$,

(23) $\bar{\mathcal{S}}_1^j \prec f_1^{-1}(\mathcal{S}_0^{j+3})$ for $j = 1,2$,

(24) $\mathcal{N}_1^j \prec \mathrm{Sd}_{2j}\,\mathcal{N}_1^0$ for $j = 1,2$.

By using Lemma [22, Proposition A.3], for the cover $\mathcal{E}_1 \equiv \left\{\mathrm{st}(x, \bar{\mathcal{S}}_1^2) : x \in |\mathcal{N}_1|\right\}$, we obtain an open cover $\mathcal{B}_1$ of $|\mathcal{N}(\mathcal{V}_0)|$ and a PL, $\mathcal{N}_1^2$-modification $r_1\colon |\mathcal{N}_1^2| \to |\mathcal{N}_1^2|$ of the identity map such that

$(12)_1$ $r_1(\mathrm{Cl}\,B)$ is compact for $B \in \mathcal{B}_1$,

$(13)_1$ $\mathrm{Cl}\,B \cup r_1(\mathrm{Cl}\,B) \subseteq E$ for some $E \in \mathcal{E}_1$.

Since b_1 is (G,n)-cohomological, from the similar argument to the proof of the necessity in Theorem 3.3 we can take the followings:

subdivision $\mathcal{N}_1^3$ of $\mathrm{Sd}_2\mathcal{N}_1^2$, locally finite open cover $\mathcal{V}_2$ of X and maps $b_2\colon X \to |\mathcal{N}(\mathcal{V}_2)|$, $f_2^*\colon |\mathcal{N}(\mathcal{V}_2)| \to |\mathcal{N}_1^3|$ such that

$(14)_2$ $\bar{\mathcal{S}}_1^3 \prec^* \mathcal{S}_1^2 \wedge \mathcal{B}_1 \wedge f_1^{-1}(\mathcal{S}_0^6)$,

$(15)_2$ $\mathcal{V}_2 \prec^* \mathcal{U}_2 \wedge b_1^{-1}(\mathcal{S}_1^3) \wedge b_0^{-1}(\mathcal{S}_0^6)$,

$(16)_2$ b_2 is $\mathcal{V}_2$-normal,

$(17)_2$ $f_2^* \circ b_2$ is $\mathcal{N}_1^3$-modification of b_1,

$(18)_2$ for each $\sigma \in \mathcal{N}(\mathcal{V}_2)$, there exists $U \in \mathrm{st}\,\mathcal{S}_1^3$ such that $b_1\left(b_2^{-1}(\sigma)\right) \cup f_2^*(\sigma) \subseteq U$,

$(19)_2$ for any triangulation M of $|\mathcal{N}(\mathcal{V}_2)|$, there exists a PL-map $p'\colon |M^{(n)}| \to |(\mathcal{N}_1^3)^{(n)}|$ such that

(i) $(p', f_2^*|_{|M^{(n)}|}) \leq \{\bar{\mathrm{st}}(\lambda, \mathcal{N}_1^3) : \lambda \in \mathcal{N}_0^3\}$,

(ii) for any map $\alpha\colon |(\mathcal{N}_1^3)^{(n)}| \to K(G,n)$, there exists an extension $\beta\colon |M^{(n+1)}| \to K(G,n)$ of $\alpha \circ p'$.

Now, by using $(19)_1$ about the triangulation $\mathcal{N}_1^3$ of $|\mathcal{N}(\mathcal{V}_1)|$, we obtain a PL-map $g_1^*\colon |(\mathcal{N}_1^3)^{(n)}| \to |(\mathcal{N}_0^3)^{(n)}|$ such that

$(25)_1$ $(g_1^*, f_1^*|_{|(\mathcal{N}_1^3)^{(n)}|}) \leq \{\bar{\mathrm{st}}(\lambda, \mathcal{N}_0^3) : \lambda \in \mathcal{N}_0^3\}$,

$(26)_1$ for any map $\alpha\colon |(\mathcal{N}_0^3)^{(n)}| \to K(G,n)$, there exists an extension $\beta\colon |(\mathcal{N}_1^3)^{(n+1)}| \to K(G,n)$ of $\alpha \circ g_1^*$.

Consider the inclusion map $i_0\colon |(\mathcal{N}_0^3)^{(n)}| \hookrightarrow |\mathcal{N}_0^3|$ and the composition

$$r_0 \circ i_0 \circ g_1^*\colon |(\mathcal{N}_1^3)^{(n)}| \to |(\mathcal{N}_0^3)^{(n)}| \hookrightarrow |\mathcal{N}_0^3| = |\mathcal{N}(\mathcal{V}_0)| \to |\mathcal{N}(\mathcal{V}_0)|.$$

The image A of the PL-map $r_0 \circ i_0 \circ g_1^*$ has dimension $\leq n$. Then we can take a $\mathcal{N}_0^3$-modification $s_0\colon A \to |(\mathcal{N}_0^3)^{(n)}|$ of the inclusion map $A \hookrightarrow |\mathcal{N}_0^3|$. Let $g_1\colon |(\mathcal{N}_1^3)^{(n)}| \to |(\mathcal{N}_0^3)^{(n)}|$ denote the composition map $s_0 \circ r_0 \circ i_0 \circ g_1^*$.

Then this has the following properties:

Claim 1.

$(9)_1$ *for each $t \in |(\mathcal{N}_1^3)^{(n)}|$, there exist $\sigma, \tau \in \mathcal{N}_0^2$ such that $f_1(t) \in \sigma$, $g_1(t) \in \tau$ and $\sigma \cap \tau \neq \emptyset$,*

$(10)_1$ *for any map $\alpha\colon |(\mathcal{N}_0^3)^{(n)}| \to K(G,n)$, there exist an extension $\beta\colon |(\mathcal{N}_1^3)^{(n+1)}| \to K(G,n)$ of $\alpha \circ g_1$,*

$(11)_1$ *for each $x \in |\mathcal{N}_1|$, $g_1\left(\mathrm{st}(x, \bar{\mathcal{S}}_1^2) \cap |(\mathcal{N}_1^3)^{(n)}|\right)$ is a Whitehead (i.e. finite) compact polyhedral subset of $|\mathcal{N}_0|$.*

Proof of Claim 1. We show the property $(9)_1$. Let $t \in |(\mathcal{N}_1^3)^{(n)}|$. By $(25)_1$, there exist $\sigma, \lambda, \tau \in \mathcal{N}_0^3$ such that $f_1^*(t) \in \sigma$, $g_1^*(t) \in \tau$ and $\sigma \cap \lambda \neq \emptyset \neq \lambda \cap \tau$. We may assume that $\lambda = |v_0, v_1|$, $v_0 \in \sigma$ and $v_1 \in \tau$.

Since j_0 is $\mathcal{N}_0^3$-modification of the identity function, we have $j_0 \circ f_1^*(t) \in \sigma$. Since f_1 is simplicial approximation of $j_0 \circ f_1^*$, we have $f_1(t) \in \sigma$.

Select $\tilde{\tau} \in \mathcal{N}_0^2$ with $\tau \subseteq \tilde{\tau}$. Since r_0 is $\mathcal{N}_0^2$-modification of the identity map, we have $r_0 \circ i_0 \circ g_1^*(t) \in \tilde{\tau}$. Further since s_0 is $\mathcal{N}_0^3$-modification of $A \hookrightarrow |\mathcal{N}_0^3|$ and $\mathcal{N}_0^3 \prec \mathcal{N}_0^2$, we have $g_1(t) = s_0 \circ r_0 \circ i_0 \circ g_1^*(t) \in \tilde{\tau}$.

Case 1. $v_1 \in (\mathcal{N}_0^2)^{(0)}$ (i.e. $v_1 \in \tilde{\tau}^{(0)}$).

By $\mathcal{N}_0^3 \prec \mathrm{Sd}_2 \mathcal{N}_0^2$, we have $v_0 \notin (\mathcal{N}_0^2)^{(0)}$. Hence, there exists $\gamma \in \mathcal{N}_0^2$ such that $|v_0, v_1| \subseteq \gamma$ and $v_0 \in \mathrm{Int}\,\gamma$. Then if $\tilde{\sigma} \in \mathcal{N}_0^2$ with $\sigma \subseteq \tilde{\sigma}$, we have $\gamma \prec \tilde{\sigma}$. Therefore we have $\tilde{\sigma} \cap \tilde{\tau} \neq \emptyset$, $f_1(t) \in \tilde{\sigma}$ and $g_1(t) \in \tilde{\tau}$.

Case 2. $v_1 \notin (\mathcal{N}_0^2)^{(0)}$.

If $v_0 \in (\mathcal{N}_0^2)^{(0)}$, the proof is similar to Case 1. Let $v_0 \notin (\mathcal{N}_0^2)^{(0)}$. By $\mathcal{N}_0^3 \prec \mathrm{Sd}_2 \mathcal{N}_0^2$, there exist $\gamma_0, \gamma_1 \in \mathcal{N}_0^2$ such that $v_0 \in \mathrm{Int}\,\gamma_0$, $v_1 \in \mathrm{Int}\,\gamma_1$ and $\gamma_0 \prec \gamma_1$ or $\gamma_1 \prec \gamma_0$. Then if $\tilde{\sigma} \in \mathcal{N}_0^2$ with $\sigma \subseteq \tilde{\sigma}$, we have $\gamma_0 \prec \tilde{\sigma}$. Similarly, we have $\gamma_1 \prec \tilde{\tau}$. Therefore we have $\tilde{\sigma} \cap \tilde{\tau} \neq \emptyset$, $f_1(t) \in \tilde{\sigma}$ and $g_1(t) \in \tilde{\tau}$.

By $g_1^* \simeq g_1$, we can see the property $(10)_1$ by the homotopy extension theorem and $(26)_1$.

We show the property $(11)_1$. First, we shall see that

$$(27) \qquad g_1^*\left(\mathrm{st}(x, \bar{\mathcal{S}}_1^2) \cap |(\mathcal{N}_1^3)^{(n)}|\right) \subseteq B \text{ for some } B \in \mathcal{B}_0.$$

Let $\mathrm{st}(x, \bar{\mathcal{S}}_1^2)$ be represented by $\bigcup\{\bar{\mathrm{st}}(v_\alpha, \mathcal{N}_1^2) : \alpha \in A\}$. There exists $\sigma_x \in \mathcal{N}_1^2$ with $x \in \mathrm{Int}\,\sigma_x$.

For each $\alpha \in A$, we choose $\sigma_\alpha \in \mathcal{N}_1^2$ such that $\sigma_x \preccurlyeq \sigma_\alpha$ and $v_\alpha \in \sigma_\alpha$. Further we select minimum and maximal dimensional simplexes $\tau_x, \tau_\alpha \in \mathcal{N}_1^0$ with $\tau_x \preccurlyeq \tau_\alpha$ respectively such that $\sigma_x \subseteq \tau_x$ and $\sigma_\alpha \subseteq \tau_\alpha$.

If $\sigma_x \subseteq \mathrm{Int}\,\tau_x$, we have $\bar{\mathrm{st}}(v_\alpha, \mathcal{N}_1^2) \subseteq \tau_\alpha$ from $v_\alpha \in \mathrm{Int}\,\tau_\alpha$. Then there exists a vertex $v \in \mathcal{N}_1^2$ such that $\bigcup_\alpha \tau_\alpha \subseteq \bar{\mathrm{st}}(v, \mathcal{N}_1^0)$. Since f_1 is the simplicial map from $\mathcal{N}_1^0$ to $\mathcal{N}_0^3$, we have $f_1\left(\bigcup_\alpha \tau_\alpha\right) \subseteq f_1(\bar{\mathrm{st}}(v, \mathcal{N}_1^0)) \subseteq \bar{\mathrm{st}}(f_1(v), \mathcal{N}_0^3)$. By the nearness between f_1 and g_1^* (see proof of $(9)_1$) and $(14)_1$, we obtain

$$(28) \qquad g_1^*\left(\mathrm{st}(x, \bar{\mathcal{S}}_1^2) \cap |(\mathcal{N}_1^3)^{(n)}|\right) \subseteq \mathrm{st}\left(\bar{\mathrm{st}}(f_1(v), \mathcal{N}_0^3), \bar{\mathcal{S}}_0^3\right) \subseteq B \text{ for some } B \in \mathcal{B}_0.$$

If $\sigma_x \cap \partial\tau_x \neq \emptyset$ and $\sigma_x \cap \mathrm{Int}\,\tau_x \neq \emptyset$, we choose a face $\tilde{\tau}_x$ with $\tilde{\tau}_x \precneqq \tau_x$ such that $\sigma_x \cap \partial\tau_x \subseteq \tilde{\tau}_x$. Then there exists a vertex $v \in \tilde{\tau}_x$ such that $\bigcup_\alpha \bar{\mathrm{st}}(v_\alpha, \mathcal{N}_1^2) \subseteq \bar{\mathrm{st}}(v, \mathcal{N}_1^0)$. Hence we have (28) in the same way.

Since $\mathrm{st}(x, \bar{\mathcal{S}}_1^2) \cap |(\mathcal{N}_1^3)^{(n)}|$ is a subpolyhedron of $|\mathcal{N}_1|$ and g_1^* is a PL-map, we see that $g_1^*\left(\mathrm{st}(x, \bar{\mathcal{S}}_1^2) \cap |(\mathcal{N}_1^3)^{(n)}|\right)$ is a subpolyhedron of $|\mathcal{N}_0|$. Then by (27) and $(12)_0$, $r_0 \circ i_0 \circ g_1^*\left(\mathrm{st}(x, \bar{\mathcal{S}}_1^2) \cap |(\mathcal{N}_1^3)^{(n)}|\right)$ is a subpolyhedron of $|\mathcal{N}_0|$ and a compact set of $|\mathcal{N}_0|_w$. Since s_0 is a PL-map, we have see the property $(11)_1$.

Now, we shall take a base for a uniformity for $|\mathcal{N}_1|$. We choose a subdivisions $\mathcal{N}_1^j$ for $j \geq 4$ of $\mathcal{N}_1$ such that

(29) $\mathcal{N}_1^{j+1} \prec \mathrm{Sd}_2 \mathcal{N}_1^j$ for $j \geq 3$,

(30) $\bar{\mathcal{S}}_1^{j+1} \prec^* \mathcal{S}_1^j$ for $j \geq 3$,

(31) $\bar{\mathcal{S}}_1^{j+1} \prec f_1^{-1}(\mathcal{S}_0^{j+4}) \wedge f_1^{*-1}(\mathcal{S}_0^{j+4}) \wedge \mathcal{F}_1^{j+4}$ for $j \geq 3$,

where $\mathcal{F}_1^{j+4}$ is defined as follows. $g_1^{-1}\left(\mathcal{S}_0^{j+4} \cap |(\mathcal{N}_0^3)^{(n)}|\right)$ is the open cover of $|(\mathcal{N}_1^3)^{(n)}|_w$. Extend it to an open cover $\mathcal{F}_1^{j+4}$ of $|\mathcal{N}_1|_w$. Then clearly the uniformity make f_1, f_1^* and g_1 uniformly continuous.

We shall show that f_1 holds the property (5). First, note that the composition

$$j_0 \circ id \circ f_1^* \colon |\mathcal{N}_1|_u \to |\mathcal{N}_0|_u \to |\mathcal{N}_0|_m \to |\mathcal{N}_0|_w,$$

where $id\colon |\mathcal{N}_0|_u \to |\mathcal{N}_0|_m$ is the identity map, is continuous.

Let K be a compact set of $|\mathcal{N}_1|_u$. There exist $\sigma_1, \ldots, \sigma_l \in \mathcal{N}_0$ such that $j_0 \circ f_1^*(K) = j_0 \circ id \circ f_1^*(K) \subseteq \sigma_1 \cup \cdots \cup \sigma_l$. Since f_1 is a simplicial approximation of $j_0 \circ f_1^*$, we have $f_1(K) \subseteq \sigma_1 \cup \cdots \cup \sigma_l$. By the continuity of f_1, $f_1(K)$ is a compact set of $|\mathcal{N}_0|_u$.

As we proceed in this work, we have $\mathcal{V}_i$, f_i^*, f_i, $\mathcal{N}_i^j$ and g_i with the properties (1)-(11).

From now on, we consider X to be the uniform space with the uniformity generated by the sequence $\{\mathcal{V}_i\}_{i=0}^\infty$ of open covers of X and $|\mathcal{N}_i|$ to be the uniform space with the uniformity generated by the sequence $\{\mathcal{S}_i^j\}_{j=0}^\infty$. Then by the construction, the topology induced by $\{\mathcal{V}_i\}_{i=0}^\infty$ and the original metric topology are identical.

We shall construct the resolution of X. The construction essentially depends on Rubin's way [22]. Hence, the detail is omitted here.

For $j \geq 0$, let $f_{j,j}$ denote the identity on $\mathcal{N}_j$ and let $f_{i,j}$ denote the composition $f_{j+1} \circ \cdots \circ f_i \colon |\mathcal{N}_i| \to |\mathcal{N}_j|$ for $i > j$.

The functions

$$b_i \colon (X, \{\mathcal{V}_i\}_{i=0}^\infty) \to (|\mathcal{N}_i|, \{\mathcal{S}_i^j\}_{j=0}^\infty)$$

and

$$f_{i+1,i} \colon (|\mathcal{N}_{i+1}|, \{\mathcal{S}_{i+1}^j\}_{j=0}^\infty) \to (|\mathcal{N}_i|, \{\mathcal{S}_i^j\}_{j=0}^\infty)$$

are uniformly continuous for $i \geq 0$. Then since the sequence $\{f_{i,j} \circ b_i\}_{i=j}^\infty$ is Cauchy in the uniform space $C(X, |\mathcal{N}_j|_u)$ with the uniformity of uniform convergence, we have a uniformly continuous, limit map

$$f_{\infty,j} \equiv \lim_{q \to \infty} f_{q,j} \circ b_q \colon (X, \{\mathcal{V}_i\}_{i=0}^\infty) \to \left(|\mathcal{N}_j|, \{\mathcal{S}_j^i\}_{i=0}^\infty\right),$$

such that

(32) $f_{\infty,j}$ is $\mathcal{N}_j^3$-modification of b_j,

(33) $(f_{\infty,j}, b_j) \leq \mathcal{S}_j^1$,

(34) $f_{\infty,j}$ is a topological irreducible (i.e. surjective) map relative to $\mathcal{N}_j^3$,

(35) $f_{i+1,i} \circ f_{\infty,i+1} = f_{\infty,i}$ for $i \geq 0$.

We consider $\prod_{i=0}^{\infty} |\mathcal{N}_i|_u$ to be the uniform space by the product uniformity. Note that $\varprojlim \{|\mathcal{N}_j|_u, f_{i+1,i}\}$ is a non-empty subspace by the property (34).

Then by (35), there exist a uniformly continuous map $f_\omega\colon X \to \varprojlim |\mathcal{N}_i|_u$ with $f_{\infty,i} = pr_i \circ f_\omega$ and especially the map f_ω is a uniformly embedding onto a dense subset $f_\omega(X)$ in $\varprojlim |\mathcal{N}_i|_u$, where $pr_i\colon \prod_{j=0}^{\infty} |\mathcal{N}_j|_u \to |\mathcal{N}_i|_u$ is the natural projection.

Let Z denote the limit of the inverse sequence $\left\{|(\mathcal{N}_i^3)^{(n)}|_u, g_{i+1,i}\right\}$. Then we consider Z to be the sub-uniform space of the uniform space $\prod_{i=0}^{\infty} |\mathcal{N}_i|_u$. Note that Z has dimension $\leq n$.

We begin with a description of the map π. For $j \geq 0$, a uniformly continuous map $\pi_j\colon Z \to \prod_{i=0}^{\infty} |\mathcal{N}_i|_u$ is defined by

$$\pi_j(\mathbf{z}) \equiv (f_{j,0}(z_j), f_{j,1}(z_j), \ldots, f_{j,j-1}(z_j), z_j, z_{j+1}, \ldots)$$

for $\mathbf{z} = (z_j) \in Z$ and let π_0 be the inclusion map. Then since the sequence $\{\pi_j\}_{j=0}^{\infty}$ is Cauchy in $C(Z, \prod_{i=0}^{\infty} |\mathcal{N}_i|_u)$, there is a uniformly continuous, limit map $\pi\colon Z \to \prod_{i=0}^{\infty} |\mathcal{N}_i|_u$. Then the map π is proper from Z onto $\varprojlim\{|\mathcal{N}_i|_u, f_{i+1,i}\}$. We must show that $\pi^{-1}(\mathbf{x})$ is a UV^{n-1}-set and the set $[\pi^{-1}(\mathbf{x}), K(G,n)]$ is trivial for $\mathbf{x} \in \varprojlim\{|\mathcal{N}_i|_u, f_{i+1,i}\}$.

For $\mathbf{x} = (x_i) \in \varprojlim\{|\mathcal{N}_i|_u, f_{i+1,i}\}$, let $\delta N(x_i)$ and $\varepsilon N(x_i)$ denote $\mathrm{st}(x_i, \bar{\mathcal{S}}_i^0)$ and $\mathrm{st}(x_i, \bar{\mathcal{S}}_i^2)$, respectively. Then we have the following properties [22]: for $\mathbf{x} = (x_i) \in \varprojlim\{|\mathcal{N}_i|_u, f_{i+1,i}\}$,

(36) $g_{i,i-1}\left(\delta N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|\right) \subseteq \varepsilon N(x_{i-1})$,

(37) $\varprojlim \left\{\varepsilon N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|, g_{i,i-1}|_{\ldots}\right\} = \pi^{-1}(\mathbf{x}) = \varprojlim \left\{\delta N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|, g_{i,i-1}|_{\ldots}\right\}$

By $\bar{\mathcal{S}}_i^2 \prec^* \mathcal{S}_i^1$, there exists $F_i \in \mathcal{S}_i^1$ such that $\mathrm{st}(x_i, \bar{\mathcal{S}}_i^2) \subseteq F_i$. Further, by $\mathcal{S}_i^1 \prec \mathcal{S}_i^0$, there is a $S \in \mathcal{S}_i^0$ such that $F_i \subseteq S$. Hence we have the contractible set F_i such that

$$\varepsilon N(x_i) \subseteq F_i \subseteq \delta N(x_i). \tag{38}$$

Claim 2. *$\pi^{-1}(\mathbf{x})$ is a UV^{n-1}-set for for $\mathbf{x} = (x_i) \in \varprojlim\{|\mathcal{N}_i|_u, f_{i+1,i}\}$.*

Proof of Claim 2. It suffices to show that the map

$$g_{i+1,i}|_{\ldots}\colon \delta N(x_{i+1}) \cap |(\mathcal{N}_{i+1}^3)^{(n)}| \to \delta N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|$$

induces a zero homomorphism of homotopy group of dimension less than n. By (36) and (38), we have

$$g_{i+1,i}\left(\delta N(x_{i+1}) \cap |(\mathcal{N}_{i+1}^3)^{(n)}|\right) \subseteq F_i \cap |(\mathcal{N}_i^3)^{(n)}| \subseteq \delta N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|.$$

Since F_i is contractible, we have

$$\pi_k\left(F_i \cap |(\mathcal{N}_i^3)^{(n)}|\right) = 0 \quad \text{for } k < n.$$

Therefore $g_{i+1,i}|_{\ldots}$ induces a zero homomorphism of homotopy group of dimension less than n.

Claim 3. $[\pi^{-1}(\mathbf{x}), K(G,n)] \approx \check{H}^n(\pi^{-1}(\mathbf{x});G)$ *is trivial for* $\mathbf{x} \in \varprojlim\{|\mathcal{N}_i|_u, f_{i+1,i}\}$.

Proof of Claim 3. By (11),(36),(37) and the continuity of Čech cohomology, we have

$$\check{H}^n(\pi^{-1}(\mathbf{x});G) \approx \varinjlim \left\{H^n\left(g_{i,i-1}(\varepsilon N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|_u);G\right), g_{i,i-1}|_{\dots}^{*}\right\}.$$

Hence it suffices to show that

$$g_{i,i-1}|_{\dots}^{*}: H^n\left(g_{i,i-1}(\varepsilon N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|);G\right) \to H^n\left(g_{i+1,i}(\varepsilon N(x_{i+1}) \cap |(\mathcal{N}_{i+1}^3)^{(n)}|);G\right)$$

is the zero homomorphism.

Let $G_{i,i-1}$ denotes $g_{i,i-1}\left(\varepsilon N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|_u\right)$. Then by (11) the subspace $G_{i,i-1}$ of $|(\mathcal{N}_{i-1}^3)^{(n)}|_u$ and the subspace $G_{i,i-1}$ of $|(\mathcal{N}_{i-1}^3)^{(n)}|_w$ is identical. Hence from now on, we may consider that $G_{i,i-1}$ is the subspace of $|(\mathcal{N}_{i-1}^3)^{(n)}|_w$.

Let $[\alpha] \in [G_{i,i-1}, K(G,n)]$. Then from $\pi_q(K(G,n)) = 0$ for $q < n$, there exists an extension $\tilde{\alpha}: |(\mathcal{N}_{i-1}^3)^{(n)}|_w \to K(G,n)$ of α. By (10), we have an extension $\beta: |(\mathcal{N}_i^3)^{(n+1)}|_w \to K(G,n)$ of $\tilde{\alpha} \circ g_{i,i-1}|_{G_{i+1,i}}$.

Since F_i is the contractible set, $F_i \cap |(\mathcal{N}_i^3)^{(n)}|_w$ is contractible in $F_i \cap |(\mathcal{N}_i^3)^{(n+1)}|_w$. Hence, there exists a homotopy $H: \left(F_i \cap |(\mathcal{N}_i^3)^{(n)}|_w\right) \times I \to F_i \cap |(\mathcal{N}_i^3)^{(n+1)}|_w$ such that H_0 is the inclusion map and H_1 is a constant map. Since $G_{i+1,i} \subseteq \varepsilon N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|_w \subseteq F_i \cap |(\mathcal{N}_i^3)^{(n)}|_w$, we can define the following compositions:

$$\begin{aligned}\tilde{H} \equiv \beta \circ i_2 \circ H \circ i_1 : G_{i+1,i} \times I \hookrightarrow \left(F_i \cap |(\mathcal{N}_i^3)^{(n)}|_w\right) \times I \to F_i \cap |(\mathcal{N}_i^3)^{(n+1)}|_w \\ \hookrightarrow |(\mathcal{N}_i^3)^{(n+1)}|_w \to K(G,n),\end{aligned}$$

where i_1 and i_2 are the inclusion maps.

Then we have $\tilde{H}_0 = \beta|_{G_{i+1,i}} = \alpha \circ g_{i,i-1}|_{G_{i+1,i}}$ and $\tilde{H}_1 =$ a constant . It completes the proof of Claim 3. Then the map

$$\pi_X \equiv \pi|_{\pi^{-1}(X)}: \pi^{-1}(X) \twoheadrightarrow X$$

is a desired one for Theorem. □

5.2. Corollary. *Let X be a metrizable space having cohomological dimension with respect to $\mathbf{Z}_p$ of less than and equal to n. Then there exist an n-dimensional metrizable space Z and a perfect* UV^{n-1}*-surjection $\pi: Z \to X$ such that for $x \in X$, the set $[\pi^{-1}(x), K(\mathbf{Z}_p,n)]$ of homotopy classes is trivial.* .

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